Problem Set 1

Here we are – the first problem set of the quarter! This problem set is designed to give you practice writing proofs on a variety of different topics. We hope that this problem set gives you a sense of what proof-based mathematics is like and helps solidify your understanding of set theory.

Before you start this problem set, please do the following:

- Sign up for Piazza so that you have an easy way to ask us questions.
- Review the office hours timetable to find good times to drop on by to ask questions.
- Review Handout #07, "Set Theory Definitions," for a refresher on key terms, definitions, and theorems about set theory that might come up in this problem set.
- Review Handout #08, "Guide to Proofs," which has advice about how to write and structure your proofs.
- Review Handout #09, "Mathematical Vocabulary," which covers mathematical phrases you may need to use in your proofs and how to use them correctly.
- Review Handout #10, "Guide to Indirect Proofs," which provides some guidance about how to set up proofs by contradiction and contrapositive.
- Review Handout #11, "Ten Techniques to Get Unstuck," for advice about how to make progress on these sorts of problems when you're not sure what to do.
- Review Handout #12, "Proofwriting Checklist," for a detailed set of criteria you should apply to your proofs before submitting them. ***We will be running this same checklist on your proofs when grading, so please be sure to look over it before submitting!***
- Review the online "Guide to $\in$ and $\subseteq$" to make sure you understand the distinction between these terms.

As always, please feel free to drop by office hours or post on Piazza if you have any questions. We're happy to help out.

Good luck, and have fun!

Checkpoint Questions Due Monday, January 15th at 2:30PM Pacific time.
Remaining Questions Due Friday, January 19th at 2:30PM Pacific time.

Write your solutions to the following checkpoint problems and submit them through GradeScope by Monday at 2:30PM Pacific time. These problems will be graded on a 0 / 1 / 2 scale. Solutions that reasonably attempt to solve all of the problems, even if the attempts are incorrect, will receive two points. Solutions that reasonably attempt some but not all of the problems will receive one point. Solutions that do not reasonably attempt any of the problems – or solutions that are submitted after the deadline – will receive zero points.

Essentially, if you've made a good, honest effort to solve all of the problems and you submit on time, you should receive two points even if your solutions contain errors.

Please make the best effort you can when solving these problems. We want the feedback we give you on your solutions to be as useful as possible, so the more time and effort you put into them, the better we'll be able to comment on your proof style and technique. We will try to get these problems returned to you with feedback on your proof style by Wednesday. Submission instructions are included in the "Problem Set Policies" handout.

Checkpoint Problem One: Finding Negations

In order to write a proof by contradiction or contrapositive, you'll need to determine the negation of one or more statements. In Friday's lecture, we talked about a few common classes of statements and how to form their negations. Using what you've learned, answer the following multiple-choice questions and ***briefly explain how you arrived at your answer***.

Which of the following is the negation of "everything that has a beginning has an end?"

A) Everything that does not have a beginning has an end.
B) Everything that has a beginning has no end.
C) There is something that has no beginning and has an end.
D) There is something that has a beginning and has no end.

Which of the following is the negation of "there is a successful person who is grateful?"

A) There is an unsuccessful person who is grateful.
B) There is a successful person who is ungrateful.
C) Every successful person is grateful.
D) Every successful person is ungrateful.
E) Every unsuccessful person is grateful.
F) Every unsuccessful person is ungrateful.

Which of the following is the negation of "if $A \subseteq B$, then $A - B = \varnothing$?"

A) If $A \subseteq B$, then $A - B = \varnothing$.
B) If $A \subseteq B$, then $A - B \neq \varnothing$.
C) If $A \not\subseteq B$, then $A - B = \varnothing$.
D) If $A \not\subseteq B$, then $A - B \neq \varnothing$.
E) There are sets A and B where $A \subseteq B$ and $A - B = \varnothing$.
F) There are sets A and B where $A \subseteq B$ and $A - B \neq \varnothing$.
G) There are sets A and B where $A \not\subseteq B$ and $A - B = \varnothing$.
H) There are sets A and B where $A \not\subseteq B$ and $A - B \neq \varnothing$.

Remember that you need to provide a justification for your answers. While it's not required, ideally you should be able to explain both why your answer is correct and why all the other answers are incorrect.

Checkpoint Problem Two: Multiples of Three

In class, we talked a fair amount about odd and even numbers, which arise when dividing numbers in half. This question generalizes the idea of "even" and "odd" to similar terms that arise when dividing by three.

An integer is called a ***multiple of three*** if it can be written as $3k$ for some integer k. An integer is ***congruent to one modulo three*** if it can be written as $3k + 1$ for some integer k, and an integer is ***congruent to two modulo three*** if it can be written as $3k + 2$ for some integer k. For each integer n, exactly one of the following is true (you don't need to prove this):

- n is a multiple of three.
- n is congruent to one modulo three.
- n is congruent to two modulo three.

We'd like you to prove this result:

For every integer n, n is a multiple of three if and only if n^2 is a multiple of three.

To do this, we'll have you prove the following two statements:

For any integer n, if n a multiple of three, then n^2 is a multiple of three.

For any integer n, if n^2 is a multiple of three, then n is a multiple of three.

We've broken this question down into a few parts.

i. Prove the first of these statements with a direct proof.

Not sure how to do that? Take a look at our proof that if n is even, then n^2 is even.

ii. Prove the second of these statements using a proof by contrapositive. Make sure that you state the contrapositive of the statement explicitly before you attempt to prove it.

As a hint, think about using a proof by cases.

iii. Prove, by contradiction, that $\sqrt{3}$ is irrational.

You may want to read over the proof that the square root of two is irrational and use it as a starting point.

The remainder of these problems are due by Friday at 2:30PM.

Problem One: Set Theory Warmup

This question is designed to help you get used to the notation and mathematical conventions surrounding sets. Consider the following sets:

$$A = \{ 1, 2, 3, 4 \}$$
$$B = \{ 2, 2, 2, 1, 4, 3 \}$$
$$C = \{ 1, \{2\}, \{\{3, 4\}\} \}$$
$$D = \{ 1, 3 \}$$
$$E = \mathbb{N}$$
$$F = \{ \mathbb{N} \}$$

Answer each of the following questions and briefly justify your answers. No proofs are necessary.

i. Which pairs of the above sets, if any, are equal to one another?

ii. Is $D \in A$? Is $D \subseteq A$?

iii. What is $A \cap C$? How about $A \cup C$? How about $A \Delta C$?

iv. What is $A - C$? How about $\{A - C\}$? Are those sets equal?

v. What is $|B|$? What is $|E|$? What is $|F|$?

vi. What is $E - A$? Express your answer in set-builder notation.

vii. Is $0 \in E$? Is $0 \in F$?

Problem Two: The Power Set Revisited

In our first lecture, we saw an operation called the power set that, given a set S, produces a set $\wp(S)$ consisting of all the subsets of the set S. Why *didn't* we introduce an operation that, given a set S, produces a set consisting of all the *elements* of S?

Problem Three: Much Ado About Nothing

It can take a bit of practice to get used to the empty set. This problem will ask you to think about a few different sets related to $\varnothing$.

Go to the CS103 website and download the starter files for Problem Set One. Unpack the files somewhere convenient and open up the bundled project. Answer each part of this question by editing the relevant resource files (they're in the `res/` directory). There's information in the top of each of the files about how to represent sets; most importantly, note that to write out the empty set, you should write `{}` rather than using the empty-set symbol. For example, the set $\{\varnothing\}$ would be written as `{{}}`.

i. Edit the file `PartI.object` so that it contains a set equal to $\varnothing \cup \{\varnothing\}$.

ii. Edit the file `PartII.object` so that it contains a set equal to $\varnothing \cap \{\varnothing\}$.

iii. Edit the file `PartIII.object` so that it contains a set equal to $\{\varnothing\} \cup \{\{\varnothing\}\}$.

iv. Edit the file `PartIV.object` so that it contains a set equal to $\{\varnothing\} \cap \{\{\varnothing\}\}$.

v. Edit the file `PartV.object` so that it contains a set equal to $\wp(\wp(\varnothing))$.

vi. Edit the file `PartVI.object` so that it contains a set equal to $\wp(\wp(\wp(\varnothing)))$.

The starter code contains a driver program you can use to see the contents of your files and confirm they're syntactically correct. Submit your answers through GradeScope under "Coding Problems for Problem Set One" by uploading these six files. You're welcome to submit as many times as you'd like.

Problem Four: Set Theory in C++

The C++ standard library contains a type called `std::set` that represents a set of elements, all of which must be of the same type. For example, the type `std::set<int>` represents a set of integers, the type `std::set<std::string>` represents a set of strings, and `std::set<std::set<int>>` is a type representing a set of sets of integers.

There are all sorts of operations you can perform on `std::set`s. For example, here's how you iterate over all the elements of a set:

```
std::set<T> mySet;
for (T elem: mySet) {
    /* … do something with the current element elem … */
}
```

Here's how you check whether a particular value is an element of a set:

```
if (mySet.count(value)) {
    /* … value ∈ mySet … */
} else {
    /* … value ∉ mySet … */
}
```

And, finally, here's how you can get the cardinality of a set:

```
size_t size = mySet.size();
```

Here, the `size_t` type is a type representing a natural number, since sets can't have negative size. (The folks who implemented the C++ standard libraries had a strong discrete math background.)

One of the major differences between the sets that we've been talking about in CS103 and the `std::set` type is that in discrete mathematics, sets can contain anything – numbers, philosophical concepts, recipes, other sets, etc. – but in C++ all objects in a set must have the same type. For the purposes of this problem, we've created a custom C++ type called `Object`. Variables of type `Object` can represent just about anything, so a `std::set<Object>` represents something pretty similar to the sets we've been studying so far.

Some `Object`s are actually just `std::set`s in disguise. If you have an `Object`, you can test whether it's actually a set by using this provided helper function:

```
bool isSet(Object o);
```

This takes in an `Object`, then returns true if that `Object` is actually a set and false otherwise. If you have an `Object` that really is a set, you can convert it to a set by using this helper function:

```
std::set<Object> asSet(Object o);
```

This function takes in an `Object` that you know happens to be a set, then returns the `std::set<Object>` that it actually is.

For example, suppose you have an `Object` that you know is really the set {1, 2, 3, 4}. You could iterate over it using this code:

```
Object reallyASet = /* … */;
for (Object x: asSet(reallyASet)) {
    /* … do something with x … */
}
```

In this problem, we'd like you to demonstrate your understanding of sets and set theory by coding up a number of functions in C++ that operate on sets. In doing so, we hope that you'll solidify your grasp of the distinctions between related concepts in set theory, such as the the ∈ and ⊆ relations and power sets.

(Continued on the next page)

Open the file `SetTheory.cpp` from the starter files. There, you'll find a bunch of stubs of functions that you'll need to implement. The provided starter code contains a test harness you can use to try out your functions. You won't need to modify any of the other C++ files bundled with the starter code.

As with Problem Three, you'll submit the code that you write through GradeScope separately from the rest of the problems on this problem set. The GradeScope autograder will get back to you with feedback about how you're doing on this problem, and you're welcome to submit as many times as you'd like.

i. Implement a function

```
bool isElementOf(Object S, Object T);
```

that takes as input two `Object`s `S` and `T`, then returns whether $S \in T$.

`S` and `T` might not be sets; you'll need to use the `isSet` and `asSet` functions appropriately.

ii. Implement a function

```
bool isSubsetOf(Object S, Object T);
```

that takes as input an object `S` and an object `T`, then returns whether $S \subseteq T$.

`S` and `T` might not be sets; use the `isSet` predicate to check whether the appropriate arguments are sets and `asSet` to get a view of them as sets.

iii. Implement a function

```
bool areDisjointSets(Object S, Object T);
```

that takes as input two objects `S` and `T`, then returns whether `S` and `T` are sets where $S \cap T = \varnothing$. (Two sets with this property are called ***disjoint***.) The input parameters `S` and `T` may or may not be sets, and if they aren't, your function should return false.

iv. Implement a function

```
bool isSingletonOf(Object S, Object T);
```

that takes as input two objects `S` and `T`, then returns whether $S = \{T\}$. Again, `S` and `T` may or may not be sets.

v. Implement a function

```
bool isElementOfPowerSet(Object S, Object T);
```

that takes as input two objects `S` and `T`, then returns whether `S` and `T` are sets and $S \in \wp(T)$. Again, `S` and `T` may or may not be sets.

As a hint, you shouldn't need to write code that computes $\wp(T)$ explicitly. See if you can find a different way to do this.

vi. Implement a function

```
bool isSubsetOfPowerSet(Object S, Object T);
```

that takes as input two objects `S` and `T`, then returns whether `S` and `T` are sets and $S \subseteq \wp(T)$. Again, `S` and `T` may or may not be sets.

vii. Implement a function

```
bool isSubsetOfDoublePowerSet(Object S, Object T);
```

that takes as input two objects `S` and `T`, then returns whether `S` and `T` are sets and $S \subseteq \wp(\wp(T))$. Again, `S` and `T` may or may not be sets.

To submit your work, upload your edited `SetTheory.cpp` file to GradeScope. You'll get immediate feedback on your score from our autograder. (Don't forget to include the files from Problem Three!)

Problem Five: Describing the World in Set Theory

The notation of set theory (e.g. ∪, ∩, ℘, ⊆, ∈, etc.) is a great tool for describing the real world. Answer each of the following questions by writing an expression using set theory notation, but ***without*** using plain English, ***without*** using set-builder notation, ***without*** introducing any new variables, and ***without*** using propositional or first-order logic (which we'll cover next week).

i. Let's have C be the set of US citizens, S the set of people who live in a US state, M be the set of all people eighteen and older, and V be the set of people who are allowed to vote in US presidential elections. Write an expression that says that every US citizen age eighteen and older who lives in a US state can vote in a US presidential election.

*Once you've written up your answer to this problem, take a minute to **type-check** it. As an example, suppose that you have the following answer:*

$$(C \in M) \cap (V \in M)$$

This expression can't possibly be right, and here's one way to see this. The expression $C \in M$ is of type boolean – either $C \in M$ is true or it isn't – and the same is true of $V \in M$. However, the intersection operator ∩ can only be applied to sets. The expression therefore contains a type error: it tries to apply an operator that only works on sets to boolean values.

ii. Suppose you're on an exciting first date. Let Y represent your hobbies and D represent your date's hobbies. Write an expression that says that you have a hobby that your date doesn't have.

You can type-check this answer in a different way. For example, suppose you came up with this expression:

$$Y \cup D$$

Here, Y and D are sets, so it's perfectly fine to write $Y \cup D$, which evaluates to an object of type set. But notice that the statement here asks you to write an expression that says "you have a hobby that your date doesn't have," and that statement is essentially of type boolean (you either do or do not have a hobby your date doesn't have). Therefore. $Y \cup D$ can't possibly be an expression with the right meaning, since the type of the expression (set) doesn't match the type of the statement (boolean).

iii. Tom Stoppard's play *Rosencrantz and Guildenstern are Dead* contains this quote in which the leader of a theater troupe discusses what sorts of plays his group is willing to put on:

> "We're more of the love, blood, and rhetoric school. Well, we can do you blood and love without the rhetoric, and we can do you blood and rhetoric without the love, and we can do you all three concurrent or consecutive. But we can't give you love and rhetoric without the blood. Blood is compulsory."

Let B be the set of all plays involving blood, L be the set of all plays involving love, and R be the set of all plays involving rhetoric. Write an expression for all plays involving at least one of blood, love, and rhetoric which also happen to include blood.

iv. In the Talking Heads song *Crosseyed and Painless*, David Byrne speaks the following lines:

"Facts are simple and facts are straight.
Facts are lazy and facts are late."

Let F be the set of all facts. Let A, B, C, and D represent the set of all things that are simple, straight, lazy, and late, respectively. Write an expression that conveys David Byrne's lyrics in the language of set theory.

v. Let's say that a ***committee*** is a group of people, which we can think of as being represented by the set of people on that committee. Let's have S represent the set of all students at Stanford and let F represent the set of all faculty at Stanford. Write an expression representing the set of all committees you can make from Stanford students and faculty that contain at least one student and at least one faculty member. You can assume no one is both a student and a faculty member.

Something to think about: how would you say "all committees made purely of students?"

Problem Six: Modular Arithmetic

Different numbers can yield the same remainder when divided by some number. For example, the numbers 1, 12, 23, and 34 all leave a remainder of one when divided by eleven. To formalize this relationship between numbers, we'll introduce a relation $\equiv_k$ that, intuitively, indicates that two numbers leave the same remainder when divided by k. For example, we'd say that $1 \equiv_{11} 12$, since both 1 and 12 leave a remainder of 1 when divided by 11, and that $8 \equiv_3 14$, since both 8 and 14 leave a remainder of 2 when divided by 3. To be more rigorous, we'll formally define $\equiv_k$. For any integer k, define $a \equiv_k b$ as follows:

We say that $a \equiv_k b$ if there exists an integer q such that $a - b = kq$

For example, $7 \equiv_3 4$, because $7 - 4 = 3 = 3 \cdot 1$, and $13 \equiv_4 5$ because $13 - 5 = 8 = 4 \cdot 2$. If $x \equiv_k y$, we say that ***x is congruent to y modulo k***, hence the terminology in the checkpoint problem. In this problem, you will prove several properties of modular congruence.

i. Prove that for any integer x and any integer k that $x \equiv_k x$.

Be careful not to assume what you need to prove. Don't start your proof by assuming there's a choice of q where $x - x = kq$ and then solving for q. If you assume there's an integer q where $x - x = kq$, you're already assuming that $x \equiv_k x$! Look at the proofs we did in lecture with odd and even numbers as an example of how to prove that there is a number with a specific property without making any unfounded assumptions.

ii. Prove that for any integers x and y and any integer k that if $x \equiv_k y$, then $y \equiv_k x$.

Keep an eye out for your variable scoping in the above proof. Make sure you introduce the variables x, y, and k before you use them. Are they chosen arbitrarily? Do they represent specific values?

iii. Prove that for any integers x, y, and z and any integer k that if $x \equiv_k y$ and $y \equiv_k z$, then $x \equiv_k z$.

The three properties you have just proven show that modular congruence is an ***equivalence relation***. Equivalence relations show up everywhere in computer science, and we'll talk about them in week three.

Problem Seven: Two Is Irrational?

In lecture, we proved that $\sqrt{2}$ is irrational, and in the checkpoint problem you proved that $\sqrt{3}$ is irrational. Below is a purported proof that $\sqrt{4}$ is irrational:

Theorem: $\sqrt{4}$ is irrational.

Proof: Assume for the sake of contradiction that $\sqrt{4}$ is rational. Then there must exist integers p and q where $q \neq 0$, where $p / q = \sqrt{4}$, and where p and q have no common factors other than 1 and -1.

Starting with $p / q = \sqrt{4}$ and squaring both sides tells us that $p^2 / q^2 = 4$. We can then cross-multiply by q^2 to see that $p^2 = 4q^2$. Since q^2 is an integer and $p^2 = 4q^2$, we see that p^2 is a multiple of four, and therefore that p is a multiple of four. This tells us $p = 4n$ for some integer n.

Since we know that $4q^2 = p^2$ and $p = 4n$, we can do some algebraic substitutions to see that $4q^2 = (4n)^2 = 16n^2$, so $q^2 = 4n^2$. Since n^2 is an integer and $q^2 = 4n^2$, we see that q^2 is a multiple of four, so q is a multiple of four as well. But since both p and q are multiples of four, we see that p and q share a common divisor other than ± 1, contradicting our initial assumption. We have reached a contradiction, so our assumption must have been incorrect. Thus $\sqrt{4}$ is irrational. ■

This proof has to be wrong, because $\sqrt{4} = 2 = {}^2/_1$, so it is indeed rational! What error does this proof make that lets it conclude $\sqrt{4}$ is irrational? Be specific.

The best way to find a flaw in a proof is to find a specific, incorrect claim made in the proof and to explain, concretely, why that claim is incorrect. Also note that your job isn't to try to "fix" the proof by explaining how you'd correct the error. We just want you to convince us you see what's wrong.

Problem Eight: Properties of Sets

Here are some claims about properties of sets. Some of them are true and some of them are false. For each true statement, write a proof that the statement is true. For each false statement, write a ***disproof*** of the statement (take a look at the Proofwriting Checklist for information about how to write a disproof.) You can use any proof techniques you'd like.

i. Prove or disprove: for all sets A, B, and C, if $A \in B$ and $B \in C$, then $A \in C$.

This is your first example of a "prove or disprove" problem. Part of the challenge of approaching a problem like this one is that you'll need to figure out whether or not the statement is even true in the first place, since if it's true you'll want to prove it and if it's false you'll want to disprove it.

Here are two strategies for approaching problems like these. First, try out a lot of examples! You'll want to get a feel for what the symbolic expression above "feels" like in practice. Second, get a sheet of scratch paper and write out both the statement and its negation. One of those statements is true, and your task is to figure out which one it is. Once you have those two statements, think about what you would need to do to prove each of them. In each case, what would you be assuming? What would you need to prove? If you can answer those questions, you can explore both options and seeing which one ends up panning out.

ii. Prove or disprove: for all sets A, B, and C, if $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.

iii. Prove or disprove: for all sets A, B, and C, if $A \subsetneq B$ and $A \subsetneq C$, then $A \subsetneq B \cap C$. (The notation $A \subsetneq B$ says that A is a ***strict subset*** of B, meaning that $A \subseteq B$ and $A \neq B$.)

iv. Prove or disprove: there exists a set A where $\wp(A) = \{A\}$.

v. Prove or disprove: for all sets A and B, if $\wp(A) = \wp(B)$, then $A = B$.

Look back at Wednesday's lecture. What's a good general way to prove that two sets are equal?

Before you turn in these proofs, be sure to read over the Proofwriting Checklist and to go one item at a time through each of your proofs. Here are a few specific things to look for:

- *Make sure that the structures of your proofs match the definitions of the relevant terms. For example, to prove that a set S is a subset of a set T, follow the pattern from lecture: pick an arbitrary $x \in S$, then prove that $x \in T$ by making specific claims about x.*
- *However, avoid restating definitions in the abstract. For example, rather than writing*

  "We know that $S \subseteq T$ if every element of S is an element of T.
  Therefore, since we know that $A \subseteq B$ and $x \in A$, we see that $x \in B$."

  instead remove that first sentence and just write something like this:

  "Since $x \in A$ and $A \subseteq B$, we see that $x \in B$."

  Whoever is reading your proof knows all the relevant definitions. They're more interested in seeing ***how those definitions interact with one another*** *than* ***what those definitions are****.*
- *Make sure you clearly indicate what each variable means and whether it's chosen arbitrarily or chosen to have a specific value. For example, in your answers, if you refer to variables like A, B, or C, you should clearly indicate whether they're chosen arbitrarily or refer to specific values.*
- *If you're talking about an arbitrary set A, it's often tempting to try to list off the elements of A by writing something like $A = \{ x_1, x_2, \ldots, x_n \}$. The problem with this approach is that by writing $A = \{ x_1, x_2, \ldots, x_n \}$, you're implicitly saying that the set A is finite, since you're claiming it only has n elements in it. This is a problem if A is an infinite set. In fact, if A is infinite, because of Cantor's theorem you can't necessarily even write $A = \{ x_1, x_2, x_3, \ldots \}$, since you might run out of natural numbers with which to name the elements of A without having listed all of them!*

Problem Nine: Piano Tuning

At a first glance, irrational numbers can seem like a purely mathematical idea without any practical applications, but, surprisingly, irrational numbers have real-world implications.

Prove that $\sqrt[12]{2}$, the twelfth root of two, is irrational. Interestingly, this result means that it's impossible to tune a piano such that every half step, perfect fifth, perfect fourth, and octave are all correct. If you're curious why this is, check out this great Minute Physics video about the different ratios that arise in music and how the twelfth root of two relates.

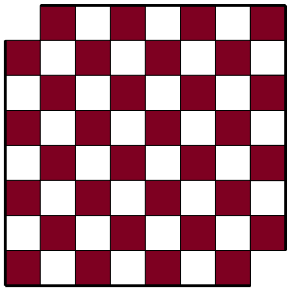
As a hint, do *not* attempt to prove this result by starting with the proof that $\sqrt{2}$ is irrational and making appropriate modifications – that will get really messy, really fast. Instead, see if you can prove the following intermediary result, and build your proof around it:

If $\sqrt[12]{2}$ is rational, then $\sqrt{2}$ is rational.

Some notes on this problem:

- *For the purposes of CS103, we've defined a rational number as a number r that can be written as $^p/_q$ for integers p and q where $q \neq 0$. For example, if you wanted to show that 1.64 is a rational number, you could just remark that it can be written as $^{164}/_{100}$ without any further elaboration, even though 164 and 100 both share 4 as a common factor. While you can always write rational numbers as a ratio of numbers with no common factors, that isn't officially part of the definition.*
- *If you want to use any properties of the rational numbers that we did not prove in class (for example, that the sum of two rational numbers is rational), you should prove those results first.*
- *You may want to read the Mathematical Prerequisites handout for a refresher on higher roots.*

Problem Ten: Tiling a Chessboard



Suppose you have a standard 8 × 8 chessboard with two opposite corners removed, as shown here. In the course notes (pages 60 - 61), there's a proof that it's impossible to tile this chessboard using 2 × 1 dominoes. This question considers what happens if you try to tile the chessboard using ***right triominoes***, L-shaped tiles that look like this:

i. Prove that it's impossible to tile an 8×8 chessboard missing two opposite corners with right triominoes.

ii. ***Prove or disprove:*** there is a natural number $n \geq 3$ where it's possible to tile an $n \times n$ chessboard missing two corners with right triominoes.

Part (ii) of this problem is another example of a prove-or-disprove type problem, and you've had plenty of practice with that from Problem Eight. So approach it the same way – grab a sheet of scratch paper, write out both the statement and its negation, work out what it is that you'd do if you wanted to prove each of those statements is true, and try a lot of examples. Explore both options and see what you find! As with part (i), drawing pictures would be a great strategy here. If you can successfully tile a board of a given size, great! You're done. If you keep running into trouble, perhaps you can spot a pattern about why that is and use that as the basis of a disproof.

Problem Eleven: Yablo's Paradox

A ***logical paradox*** is a statement that results in a contradiction regardless of whether it's true or false. One of the simplest paradoxes is the ***Liar's paradox***, which is the following:

This statement is false.

If the above statement is true, then by its own admission, it must be false – a contradiction! On the other hand, if the above statement is false, then the statement "This statement is false" is false, and therefore the statement "This statement is false" is true – a contradiction! Since this statement results in a contradiction regardless of whether it's true or false, it's a paradox.

Paradoxes often arise as a result of self-reference. In the Liar's Paradox, the paradox arises because the statement directly refers to itself. However, it's not the only paradox that can arise from self-reference. This problem explores a paradox called ***Yablo's paradox***.

Consider the following collection of infinitely many statements numbered S_0, S_1, S_2, …, where there is a statement S_n for each natural number n. These statements are ordered in a list as follows:

(S_0): All statements in this list after this one are false.
(S_1): All statements in this list after this one are false.
(S_2): All statements in this list after this one are false.
…

More generally, for each $n \in \mathbb{N}$, the statement (S_n) is

(S_n): All statements in this list after this one are false.

Surprisingly, the interplay between these statements makes every statement in the list a paradox.

i. Prove that every statement in this list is a paradox.

Some hints on this problem:

- *We've asked you to prove a universal statement (every element in this list is a paradox). What is the general template for proving a universal statement?*
- *Split your proof into two parts. First, show you get a contradiction if any of the statements in the list are true. Then, show you get a contradiction if any of the statements in the list are false.*
- *You should implicitly assume, as we've been doing in class, that every statement is either true or false. You don't need to worry about statements that are neither true nor false or statements that are simultaneously true and false.*
- *How do you negate a universally-quantified statement?*

Now, consider the following modification to this list. Instead of infinitely many statements, suppose that there are "only" 10,000,000,000 statements. Specifically, suppose we have these statements:

(T_0): All statements in this list after this one are false.
(T_1): All statements in this list after this one are false.
(T_2): All statements in this list after this one are false.
…
($T_{9,999,999,999}$): All statements in this list after this one are false.

There's still a lot of statements here, but not infinitely many of them. Interestingly, these statements are all perfectly consistent with one another and do not result in any paradoxes.

ii. For each statement in the above list, determine whether it's true or false and explain why your choices are consistent with one another.

Going forward, don't worry about paradoxical statements in CS103. We won't talk about any more statements like these. ☺

Optional Fun Problem: The Mouse and the Cheese (1 Point Extra Credit)[*]

On each problem set, we'll provide an optional fun problem for extra credit. When we compute final grades at the end of the quarter, we compute the grading curve without any extra credit factored in, then recompute grades a second time to factor in extra credit. This way, you're not at any disadvantage if you decide not to work through these problems. If you do complete the extra credit problems, you may get a slight boost to your overall grade. As a matter of course policy, we don't provide any hints on the extra credit problems – after all, they're supposed to be challenge problems! However, we're happy to chat about them after the problem sets come due.

Suppose that you have a 3" × 3" × 3" cube of cheese subdivided into twenty-seven 1" × 1" × 1" smaller cubes of cheese. A mouse wants to eat the entire cube of cheese and does so as follows: she first picks any small cube to eat first, then moves to an adjacent small cube of cheese (i.e. a cube that shares a face with the cube that was just eaten) to eat next, then repeats this process.

Prove that the mouse can't eat the centermost cube of cheese last.

[*] Adapted from Problem 4E of *A Course in Combinatorics, Second Edition* by Lint and Wilson.